Exponents and Surds

[1] EXPONENTS OR INDEX NOTATION

The use of **exponents**, also called **powers** or **indices**, allows us to write products of factors and also to write very large or very small numbers quickly.

We have seen previously that $2 \times 2 \times 2 \times 2 \times 2$, can be written as 2^5.

2^5 reads "two to the power of five" or "two with index five".

In this case 2 is the **base** and 5 is the **exponent**, **power** or **index**.

We say that 2^5 is written in **exponent** or **index notation**.

$$2^5$$

exponent, power or index ← 5

base ← 2

Example 1

Find the integer equal to: **a** 3^4 **b** $2^4 \times 3^2 \times 7$

a
$$\begin{aligned} & 3^4 \\ &= 3 \times 3 \times 3 \times 3 \\ &= 9 \times 9 \\ &= 81 \end{aligned}$$

b
$$\begin{aligned} & 2^4 \times 3^2 \times 7 \\ &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \\ &= 16 \times 9 \times 7 \\ &= 1008 \end{aligned}$$

Example 2

Write as a product of prime factors in index form:

a 144 **b** 4312

a

2	144
2	72
2	36
2	18
3	9
3	3
	1

$\therefore \quad 144 = 2^4 \times 3^2$

b

2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

$\therefore \quad 4312 = 2^3 \times 7^2 \times 11$

Historical note

Nicomachus of Gerasa lived around 100 AD. He discovered an interesting number pattern involving cubes and sums of odd numbers:

$$1 = 1^3$$
$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3 \quad \text{etc.}$$

NEGATIVE BASES

So far we have only considered **positive** bases raised to a power.

We will now briefly look at **negative** bases. Consider the statements below:

$$\begin{aligned}
(-1)^1 &= -1 \\
(-1)^2 &= -1 \times -1 = 1 \\
(-1)^3 &= -1 \times -1 \times -1 = -1 \\
(-1)^4 &= -1 \times -1 \times -1 \times -1 = 1
\end{aligned}
\qquad
\begin{aligned}
(-2)^1 &= -2 \\
(-2)^2 &= -2 \times -2 = 4 \\
(-2)^3 &= -2 \times -2 \times -2 = -8 \\
(-2)^4 &= -2 \times -2 \times -2 \times -2 = 16
\end{aligned}$$

From the pattern above it can be seen that:

- a **negative** base raised to an **odd** power is **negative**
- a **negative** base raised to an **even** power is **positive**.

Example 3

Evaluate:

a $(-2)^4$ **b** -2^4 **c** $(-2)^5$ **d** $-(-2)^5$

a $(-2)^4$
$= 16$

b -2^4
$= -1 \times 2^4$
$= -16$

c $(-2)^5$
$= -32$

d $-(-2)^5$
$= -1 \times (-2)^5$
$= -1 \times -32$
$= 32$

▪ EXPONENT OR INDEX LAWS

Notice that:

- $2^3 \times 2^4 = 2 \times 2 \times 2 \;\times\; 2 \times 2 \times 2 \times 2 = 2^7$
- $\dfrac{2^5}{2^2} = \dfrac{2 \times 2 \times 2 \times \cancel{2 \times 2}^{\,1}}{\cancel{2 \times 2}_{\,1}} = 2^3$
- $(2^3)^2 = 2 \times 2 \times 2 \;\times\; 2 \times 2 \times 2 = 2^6$
- $(3 \times 5)^2 = 3 \times 5 \times 3 \times 5 = 3 \times 3 \times 5 \times 5 = 3^2 5^2$
- $\left(\dfrac{2}{5}\right)^3 = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{2}{5} = \dfrac{2 \times 2 \times 2}{5 \times 5 \times 5} = \dfrac{2^3}{5^3}$

These examples can be generalised to the exponent or index laws:

- $a^m \times a^n = a^{m+n}$ — To **multiply** numbers with the **same base**, keep the base and **add** the indices.
- $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$ — To **divide** numbers with the **same base**, keep the base and **subtract** the indices.
- $(a^m)^n = a^{m \times n}$ — When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
- $(ab)^n = a^n b^n$ — The power of a product is the product of the powers.
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$ — The power of a quotient is the quotient of the powers.

Example 5

Simplify using the laws of indices:

a $2^3 \times 2^2$ **b** $x^4 \times x^5$

a $2^3 \times 2^2 = 2^{3+2}$
$= 2^5$
$= 32$

b $x^4 \times x^5 = x^{4+5}$
$= x^9$

To multiply, keep the base and add the indices.

To divide, keep the base and subtract the indices.

Example 6

Self Tutor

Simplify using the index laws: **a** $\dfrac{3^5}{3^3}$ **b** $\dfrac{p^7}{p^3}$

a $\dfrac{3^5}{3^3} = 3^{5-3}$

$= 3^2$

$= 9$

b $\dfrac{p^7}{p^3} = p^{7-3}$

$= p^4$

Example 7

Simplify using the index laws:

a $(2^3)^2$ **b** $(x^4)^5$

a $(2^3)^2$

$= 2^{3\times 2}$

$= 2^6$

$= 64$

b $(x^4)^5$

$= x^{4\times 5}$

$= x^{20}$

To raise a power to a power, keep the base and multiply the indices.

Each factor within the brackets has to be raised to the power outside them.

Example 8

Remove the brackets of:

a $(3a)^2$

b $\left(\dfrac{2x}{y}\right)^3$

a
$$\begin{aligned} &(3a)^2 \\ &= 3^2 \times a^2 \\ &= 9a^2 \end{aligned}$$

b
$$\begin{aligned} &\left(\frac{2x}{y}\right)^3 \\ &= \frac{2^3 \times x^3}{y^3} \\ &= \frac{8x^3}{y^3} \end{aligned}$$

Example 9

Express the following in simplest form, without brackets:

a $(3a^3b)^4$

b $\left(\dfrac{x^2}{2y}\right)^3$

a
$$\begin{aligned} &(3a^3b)^4 \\ &= 3^4 \times (a^3)^4 \times b^4 \\ &= 81 \times a^{3\times 4} \times b^4 \\ &= 81a^{12}b^4 \end{aligned}$$

b
$$\begin{aligned} &\left(\frac{x^2}{2y}\right)^3 \\ &= \frac{(x^2)^3}{2^3 \times y^3} \\ &= \frac{x^{2\times 3}}{8 \times y^3} \\ &= \frac{x^6}{8y^3} \end{aligned}$$

▪ ZERO AND NEGATIVE INDICES

Consider $\dfrac{2^3}{2^3}$ which is obviously 1.

Using the exponent law for division, $\dfrac{2^3}{2^3} = 2^{3-3} = 2^0$

We therefore conclude that $2^0 = 1$.

In general, we can state the **zero index law**: $\boldsymbol{a^0 = 1}$ **for all** $\boldsymbol{a \neq 0}$.

Now consider $\dfrac{2^4}{2^7}$ which is $\dfrac{\cancel{2 \times 2 \times 2 \times 2}^{\,1}}{2 \times 2 \times 2 \times \cancel{2 \times 2 \times 2 \times 2}_{\,1}} = \dfrac{1}{2^3}$

Using the exponent law of division, $\dfrac{2^4}{2^7} = 2^{4-7} = 2^{-3}$

Consequently, $2^{-3} = \dfrac{1}{2^3}$, which means that 2^{-3} and 2^3 are **reciprocals** of each other.

In general, we can state the **negative index law**:

> If a is any non-zero number and n is an integer, then $\boldsymbol{a^{-n} = \dfrac{1}{a^n}}$.
>
> This means that a^n and a^{-n} are **reciprocals** of one another.
>
> In particular notice that $\boldsymbol{a^{-1} = \dfrac{1}{a}}$.

Using the negative index law,

$$\begin{aligned}\left(\tfrac{2}{3}\right)^{-4} &= \frac{1}{\left(\tfrac{2}{3}\right)^4} \\ &= 1 \div \frac{2^4}{3^4} \\ &= 1 \times \frac{3^4}{2^4} \\ &= \left(\tfrac{3}{2}\right)^4\end{aligned}$$

So, in general we can see that: $\left(\dfrac{\boldsymbol{a}}{\boldsymbol{b}}\right)^{-n} = \left(\dfrac{\boldsymbol{b}}{\boldsymbol{a}}\right)^{n}$ provided $a \neq 0$, $b \neq 0$.

Example 10

Simplify, giving answers in simplest rational form:

a 7^0 **b** 3^{-2} **c** $3^0 - 3^{-1}$ **d** $\left(\frac{5}{3}\right)^{-2}$

a $7^0 = 1$

b $3^{-2} = \dfrac{1}{3^2} = \frac{1}{9}$

c $3^0 - 3^{-1} = 1 - \frac{1}{3} = \frac{2}{3}$

d $\left(\frac{5}{3}\right)^{-2} = \left(\frac{3}{5}\right)^{2} = \frac{9}{25}$

▪ STANDARD FORM

Consider the pattern alongside. Notice that each time we divide by 10, the **exponent** or **power** of 10 decreases by one.

$$\begin{array}{rcl} & 10\,000 = 10^4 & \\ \div 10 & & -1 \\ & 1000 = 10^3 & \\ \div 10 & & -1 \\ & 100 = 10^2 & \\ \div 10 & & -1 \\ & 10 = 10^1 & \\ \div 10 & & -1 \\ & 1 = 10^0 & \\ \div 10 & & -1 \\ & \frac{1}{10} = 10^{-1} & \\ \div 10 & & -1 \\ & \frac{1}{100} = 10^{-2} & \\ \div 10 & & -1 \\ & \frac{1}{1000} = 10^{-3} & \end{array}$$

We can use this pattern to simplify the writing of very large and very small numbers.

For example,

$$\begin{aligned} &5\,000\,000 \\ &= 5 \times 1\,000\,000 \\ &= 5 \times 10^6 \end{aligned}$$

and

$$\begin{aligned} &0.000\,003 \\ &= \frac{3}{1\,000\,000} \\ &= \frac{3}{1} \times \frac{1}{1\,000\,000} \\ &= 3 \times 10^{-6} \end{aligned}$$

STANDARD FORM

Standard form (or **scientific notation**) involves writing any given number as *a number between* 1 *and* 10, multiplied by an *integer power of* 10,

i.e., $a \times 10^n$ where $1 \leqslant a < 10$ and $n \in \mathbb{Z}$.

Example 11

Write in standard form: **a** 37 600 **b** 0.000 86

a $37\,600 = 3.76 \times 10\,000$ {shift decimal point 4 places to the left and $\times 10\,000$}

$= 3.76 \times 10^4$

b $0.000\,86 = 8.6 \div 10^4$ {shift decimal point 4 places to the right and $\div 10\,000$}

$= 8.6 \times 10^{-4}$

Example 12

Write as an ordinary number:

a 3.2×10^2 **b** 5.76×10^{-5}

a 3.2×10^2

$= 3.20 \times 100$

$= 320$

b 5.76×10^{-5}

$= 000005.76 \div 10^5$

$= 0.000\,057\,6$

Example 13

Simplify the following, giving your answer in standard form:

a $(5 \times 10^4) \times (4 \times 10^5)$ **b** $(8 \times 10^5) \div (2 \times 10^3)$

a $(5 \times 10^4) \times (4 \times 10^5)$

$= 5 \times 4 \times 10^4 \times 10^5$

$= 20 \times 10^{4+5}$

$= 2 \times 10^1 \times 10^9$

$= 2 \times 10^{10}$

b $(8 \times 10^5) \div (2 \times 10^3)$

$= \dfrac{8 \times 10^5}{2 \times 10^3}$

$= \frac{8}{2} \times 10^{5-3}$

$= 4 \times 10^2$

To help write numbers in standard form:

- If the original number is > 10, the power of 10 is **positive** (+).
- If the original number is < 1, the power of 10 is **negative** (−).
- If the original number is between 1 and 10, leave it as it is and multiply it by 10^0.

[2] SURDS

For the remainder of this chapter we consider **surds** and **radicals**, which are numbers that are written using the **radical** or **square root sign** $\sqrt{}$.

Surds and radicals occur frequently in mathematics, often as solutions to equations involving squared terms. We will see a typical example of this in **Chapter 8** when we study Pythagoras' theorem.

RATIONAL AND IRRATIONAL RADICALS

Some radicals are rational, but most are irrational.

For example, some rational radicals include:

$$\sqrt{1} = \sqrt{1^2} = 1 \quad \text{or} \quad \tfrac{1}{1}$$
$$\sqrt{4} = \sqrt{2^2} = 2 \quad \text{or} \quad \tfrac{2}{1}$$
$$\sqrt{\tfrac{1}{4}} = \sqrt{\left(\tfrac{1}{2}\right)^2} = \tfrac{1}{2}$$

Two examples of irrational radicals are $\sqrt{2} \approx 1.414\,214$

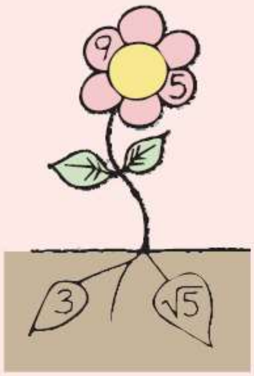
and $\sqrt{3} \approx 1.732\,051$.

Strictly speaking, a **surd** is an **irrational radical**. However, in this and many other courses, the term *surd* is used to describe *any* radical. It is reasonable to do so because the properties of surds and radicals are the same.

Historical note

The name **surd** and the **radical** sign $\sqrt{}$ both had a rather absurd past. Many centuries after Pythagoras, when the Golden Age of the Greeks was past, the writings of the Greeks were preserved, translated, and extended by Arab mathematicians.

The Arabs thought of a square number as growing out of its roots. The roots had to be extracted. The Latin word for "root" is **radix**, from which we get the words **radical** and radish! The printed symbol for radix was first **R**, then **r**, which was copied by hand as $\sqrt{}$.



The word **surd** actually came about because of an error of translation by the Arab mathematician Al-Khwarizmi in the 9th century AD. The Greek word **a-logos** means "irrational" but also means "deaf". So, the Greek **a-logos** was interpreted as "deaf" which in Latin is **surdus**. Hence to this day, **irrational radicals** like $\sqrt{2}$ are called **surds**.

BASIC OPERATIONS WITH SURDS

We have seen square roots and cube roots in previous courses. We can use their properties to help with some simplifications.

Example 15

Simplify:

a $(\sqrt{5})^2$ **b** $\left(\frac{1}{\sqrt{5}}\right)^2$

a $(\sqrt{5})^2$
$= \sqrt{5} \times \sqrt{5}$
$= 5$

b $\left(\frac{1}{\sqrt{5}}\right)^2$
$= \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}}$
$= \frac{1}{5}$

Example 16

Simplify:

a $(2\sqrt{5})^3$ **b** $-2\sqrt{5} \times 3\sqrt{5}$

a $(2\sqrt{5})^3$
$= 2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$
$= 2 \times 2 \times 2 \times \sqrt{5} \times \sqrt{5} \times \sqrt{5}$
$= 8 \times 5 \times \sqrt{5}$
$= 40\sqrt{5}$

b $-2\sqrt{5} \times 3\sqrt{5}$
$= -2 \times 3 \times \sqrt{5} \times \sqrt{5}$
$= -6 \times 5$
$= -30$

ADDING AND SUBTRACTING SURDS

'Like surds' can be added and subtracted in the same way as 'like terms' in algebra.

Consider $2\sqrt{3}+4\sqrt{3}$, which has the same form as $2x+4x$.

If we interpret this as 2 'lots' of $\sqrt{3}$ plus 4 'lots' of $\sqrt{3}$, we have 6 'lots' of $\sqrt{3}$.

So, $2\sqrt{3}+4\sqrt{3}=6\sqrt{3}$, and we can compare this with $2x+4x=6x$.

Example 17

Simplify:

a $3\sqrt{2}+4\sqrt{2}$

b $5\sqrt{3}-6\sqrt{3}$

a $3\sqrt{2}+4\sqrt{2}$
$=7\sqrt{2}$

{Compare: $3x+4x=7x$}

b $5\sqrt{3}-6\sqrt{3}$
$=-1\sqrt{3}$
$=-\sqrt{3}$

{Compare: $5x-6x=-x$}

▪ PROPERTIES OF SURDS

Discovery — Properties of surds

Notice that $\sqrt{4\times 9}=\sqrt{36}=6$ and $\sqrt{4}\times\sqrt{9}=2\times 3=6$, which suggests that $\sqrt{4}\times\sqrt{9}=\sqrt{4\times 9}$.

Also, $\sqrt{\frac{36}{4}}=\sqrt{9}=3$ and $\frac{\sqrt{36}}{\sqrt{4}}=\frac{6}{2}=3$, which suggests that $\frac{\sqrt{36}}{\sqrt{4}}=\sqrt{\frac{36}{4}}$.

What to do:

Test the following possible properties or rules for surds by substituting different values of a and b. Use your calculator to evaluate the results.

1. $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ for all $a\geqslant 0, b\geqslant 0$.
2. $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ for all $a\geqslant 0, b>0$.
3. $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ for all $a\geqslant 0, b\geqslant 0$.
4. $\sqrt{a-b}=\sqrt{a}-\sqrt{b}$ for all $a\geqslant 0, b\geqslant 0$.

You should have discovered the following properties of surds:

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ for $a \geqslant 0$, $b \geqslant 0$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0$, $b > 0$

However, in general it is not true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ or that $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$.

Example 18

Write in simplest form:

a $\sqrt{3} \times \sqrt{2}$ **b** $2\sqrt{5} \times 3\sqrt{2}$

a
$$\begin{aligned} &\sqrt{3} \times \sqrt{2} \\ &= \sqrt{3 \times 2} \\ &= \sqrt{6} \end{aligned}$$

b
$$\begin{aligned} &2\sqrt{5} \times 3\sqrt{2} \\ &= 2 \times 3 \times \sqrt{5} \times \sqrt{2} \\ &= 6 \times \sqrt{5 \times 2} \\ &= 6\sqrt{10} \end{aligned}$$

Example 19

Simplify: **a** $\dfrac{\sqrt{32}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{12}}{2\sqrt{3}}$

a
$$\begin{aligned} &\frac{\sqrt{32}}{\sqrt{2}} \\ &= \sqrt{\tfrac{32}{2}} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

b
$$\begin{aligned} &\frac{\sqrt{12}}{2\sqrt{3}} \\ &= \tfrac{1}{2}\sqrt{\tfrac{12}{3}} \quad \left\{\text{using } \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}\right\} \\ &= \tfrac{1}{2}\sqrt{4} \\ &= \tfrac{1}{2} \times 2 \\ &= 1 \end{aligned}$$

Example 20

Write $\sqrt{32}$ in the form $k\sqrt{2}$.

$$\begin{aligned}
&\sqrt{32} \\
&= \sqrt{16 \times 2} \\
&= \sqrt{16} \times \sqrt{2} \qquad \{\text{using } \sqrt{ab} = \sqrt{a} \times \sqrt{b}\} \\
&= 4\sqrt{2}
\end{aligned}$$

SIMPLEST SURD FORM

A surd is in **simplest form** when the number under the radical sign is the smallest integer possible.

Example 21

Write $\sqrt{28}$ in simplest surd form.

$$\begin{aligned}
&\sqrt{28} \\
&= \sqrt{4 \times 7} \quad \{4 \text{ is the largest perfect square factor of } 28\} \\
&= \sqrt{4} \times \sqrt{7} \\
&= 2\sqrt{7}
\end{aligned}$$

▪ MULTIPLICATION OF SURDS

The rules for expanding brackets involving surds are identical to those for ordinary algebra.

We can thus use:

$$\begin{aligned}
a(b+c) &= ab + ac \\
(a+b)(c+d) &= ac + ad + bc + bd \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a-b)^2 &= a^2 - 2ab + b^2 \\
(a+b)(a-b) &= a^2 - b^2
\end{aligned}$$

Example 22

Expand and simplify:

a $\sqrt{2}(\sqrt{2}+\sqrt{3})$ **b** $\sqrt{3}(6-2\sqrt{3})$

a
$$\begin{aligned}&\sqrt{2}(\sqrt{2}+\sqrt{3})\\&=\sqrt{2}\times\sqrt{2}\ +\ \sqrt{2}\times\sqrt{3}\\&=2+\sqrt{6}\end{aligned}$$

b
$$\begin{aligned}&\sqrt{3}(6-2\sqrt{3})\\&=(\sqrt{3})(6+-2\sqrt{3})\\&=(\sqrt{3})(6)\ +\ (\sqrt{3})(-2\sqrt{3})\\&=6\sqrt{3}+-6\\&=6\sqrt{3}-6\end{aligned}$$

Example 23

Expand and simplify:

a $-\sqrt{2}(\sqrt{2}+3)$ **b** $-\sqrt{3}(7-2\sqrt{3})$

a
$$\begin{aligned}&-\sqrt{2}(\sqrt{2}+3)\\&=-\sqrt{2}\times\sqrt{2}\ +\ -\sqrt{2}\times 3\\&=-2-3\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&-\sqrt{3}(7-2\sqrt{3})\\&=(-\sqrt{3})(7-2\sqrt{3})\\&=(-\sqrt{3})(7)\ +\ (-\sqrt{3})(-2\sqrt{3})\\&=-7\sqrt{3}+6\end{aligned}$$

Example 24

Expand and simplify: $(3-\sqrt{2})(4+2\sqrt{2})$

$$\begin{aligned}&(3-\sqrt{2})(4+2\sqrt{2})\\&=(3-\sqrt{2})(4)+(3-\sqrt{2})(2\sqrt{2})\\&=12-4\sqrt{2}+6\sqrt{2}-4\\&=8+2\sqrt{2}\end{aligned}$$

Example 25

Expand and simplify:

a $(\sqrt{3}+2)^2$

b $(\sqrt{3}-\sqrt{7})^2$

a
$$\begin{aligned}&(\sqrt{3}+2)^2\\ &=(\sqrt{3})^2 + 2\times\sqrt{3}\times 2 + 2^2\\ &=3+4\sqrt{3}+4\\ &=7+4\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}&(\sqrt{3}-\sqrt{7})^2\\ &=(\sqrt{3})^2 - 2\times\sqrt{3}\times\sqrt{7} + (\sqrt{7})^2\\ &=3-2\sqrt{21}+7\\ &=10-2\sqrt{21}\end{aligned}$$

Example 26

Expand and simplify:

a $(3+\sqrt{2})(3-\sqrt{2})$

b $(2\sqrt{3}-5)(2\sqrt{3}+5)$

a
$$\begin{aligned}&(3+\sqrt{2})(3-\sqrt{2})\\ &=3^2-(\sqrt{2})^2\\ &=9-2\\ &=7\end{aligned}$$

b
$$\begin{aligned}&(2\sqrt{3}-5)(2\sqrt{3}+5)\\ &=(2\sqrt{3})^2-5^2\\ &=(4\times 3)-25\\ &=12-25\\ &=-13\end{aligned}$$

▪ DIVISION BY SURDS

When an expression involves division by a surd, we can write the expression with an **integer denominator** which does **not** contain surds.

If the denominator contains a simple surd such as $\sqrt{a}$ then we use the rule $\sqrt{a}\times\sqrt{a}=a$.

For example: $\dfrac{6}{\sqrt{3}}$ can be written as $\dfrac{6}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$ since we are really just multiplying the original fraction by 1.

$\dfrac{6}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$ then simplifies to $\dfrac{6\sqrt{3}}{3}$ or $2\sqrt{3}$.

Example 27

Express with integer denominator: **a** $\frac{7}{\sqrt{3}}$ **b** $\frac{10}{\sqrt{5}}$ **c** $\frac{10}{2\sqrt{2}}$

a
$$\frac{7}{\sqrt{3}}$$
$$= \frac{7}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{7\sqrt{3}}{3}$$

b
$$\frac{10}{\sqrt{5}}$$
$$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{10}{5}\sqrt{5}$$
$$= 2\sqrt{5}$$

c
$$\frac{10}{2\sqrt{2}}$$
$$= \frac{10}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{10\sqrt{2}}{4}$$
$$= \frac{5\sqrt{2}}{2}$$

If the denominator has the form $a + \sqrt{b}$ then we can remove the surd from the denominator by multiplying both the numerator and the denominator by its **radical conjugate** $a - \sqrt{b}$. This produces a rational denominator, so the process is called **rationalisation** of the denominator.

Example 28

Express $\frac{1}{3+\sqrt{2}}$ with integer denominator.

$$\frac{1}{3+\sqrt{2}} = \left(\frac{1}{3+\sqrt{2}}\right)\left(\frac{3-\sqrt{2}}{3-\sqrt{2}}\right)$$
$$= \frac{3-\sqrt{2}}{3^2-(\sqrt{2})^2} \qquad \{\text{using} \quad (a+b)(a-b) = a^2 - b^2\}$$
$$= \frac{3-\sqrt{2}}{7}$$

We are really multiplying by one, which does not change the value of the original expression.

Example 29

Write $\dfrac{1-2\sqrt{3}}{1+\sqrt{3}}$ in simplest form.

$$\frac{1-2\sqrt{3}}{1+\sqrt{3}} = \left(\frac{1-2\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$

$$= \frac{1-\sqrt{3}-2\sqrt{3}+6}{1-3}$$

$$= \frac{7-3\sqrt{3}}{-2}$$

$$= \frac{3\sqrt{3}-7}{2}$$

Discovery — Continued square roots

$X = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+......}}}}}$ is an example of a **continued square root**.

Some continued square roots have actual values which are integers.

$$\sqrt{2} \approx 1.41421$$

$$\sqrt{2+\sqrt{2}} \approx 1.84776$$

$$\sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.96157\,.$$